Quadratic function

1

If the functions $f(x)$ and $g(x)$ are $f(x)=3x-1$ and $g(x)=-2x^2+4x$, find the following values.

(1) $f(0)$ (2) $f\left(-\frac{1}{3}\right)$ (3) $f(3a)$

(4) $g(2)$ (5) $g\left(\frac{1}{2}\right)$ (6) $g(a-1)$

solution

From $f(x)=3x-1$

(1) $f(0)=3\cdot0-1=\mathbf{-1}$

(2) $f\left(-\frac{1}{3}\right)=3\cdot\left(-\frac{1}{3}\right)-1=-1-1=\mathbf{-2}$

(3) $f(3a)=3\cdot3a-1=\mathbf{9a-1}$

From $g(x)=-2x^2+4x$

(4) $g(2)=-2\cdot2^2+4\cdot2=-8+8=\mathbf{0}$

(5) $g\left(\frac{1}{2}\right)=-2\cdot\left(\frac{1}{2}\right)^2+4\cdot\frac{1}{2}=-\frac{1}{2}+2=\frac{-1+4}{2}=\mathbf{\frac{3}{2}}$

(6) $g(a-1)=-2(a-1)^2+4(a-1)=-2(a^2-2a+1)+4a-4=-2a^2+4a-2+4a-4=\mathbf{-2a^2+8a-6}$

2

Find the value range of the following function.

(1) $y=3x+1 \quad (-2 \leqq x \leqq 0)$

(2) $y=-\frac{1}{3}x-2 \quad (-3 \leqq x \leqq 1)$

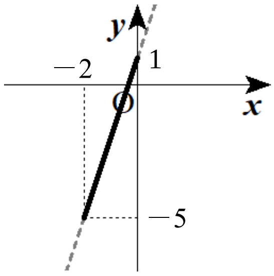
solution

(1) $y=3x+1 \quad (-2 \leqq x \leqq 0)$

The graph of the above function is shown in the figure on the right.

Therefore, the value range is

$$\mathbf{-5 \leqq y \leqq 1}.$$

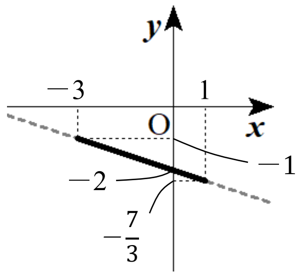


(2) $y=-\frac{1}{3}x-2 \quad (-3 \leqq x \leqq 1)$

The graph of the above function is shown in the figure on the right.

Therefore, the value range is

$$\mathbf{-\frac{7}{3} \leqq y \leqq -1}.$$



3

Answer how the graphs of the following quadratic functions are each parallel shifts of the graph of the quadratic function $y=2x^2$. Sketch the graph of each and find its axis and vertex.

(1) $y=2x^2-1$ (2) $y=2(x-2)^2$ (3) $y=2(x+1)^2-3$

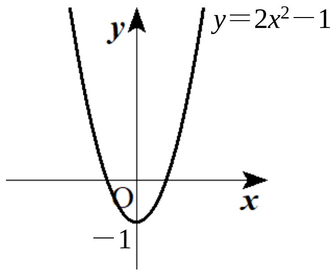
solution

(1) The parabola $y=2x^2-1$ is the parabola $y=2x^2$

shifted parallel along the y-axis direction by -1.

The axis is the straight line $x=0$ (y-axis), the vertex is $(0, -1)$,

and the graph is shown in the figure on the right.

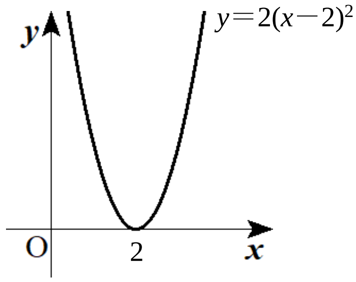


(2) The parabola $y=2(x-2)^2$ is the parabola $y=2x^2$

shifted parallel along the x-axis direction by 2.

The axis is the straight line $x=2$, the vertex is $(2, 0)$,

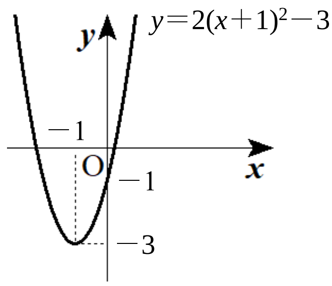
and the graph is shown in the figure on the right.



(3) The parabola $y=2(x+1)^2-3$ is expressed as parabola $y=2\{x-(-1)\}^2-3$, which means that the parabola $y=2x^2$ is **shifted parallel along the x-axis direction by -1 and along the y-axis direction by -3.**

The axis is the straight line $x=-1$, the vertex is $(-1, -3)$,

and the graph is shown in the figure on the right.



4

(1) Sketch the quadratic function $y=-3x^2-2x+1$ and find its axis and vertex.

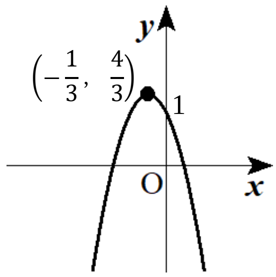
(2) When the vertices of two parabolas $y=x^2-8x$ and $y=-\frac{1}{2}x^2+ax-3b$ coincide, find the values of the constants a and b.

solution

(1) $y=-3x^2-2x+1=-3\left(x^2+\frac{2}{3}x\right)+1=-3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)+1$

$=-3\left\{\left(x+\frac{1}{3}\right)^2-\frac{1}{9}\right\}+1=-3\left(x+\frac{1}{3}\right)^2+\frac{1}{3}+1$

$=-3\left(x+\frac{1}{3}\right)^2+\frac{4}{3}$



The axis is **the line $x=-\frac{1}{3}$**, **the vertex is $\left(-\frac{1}{3},\ \frac{4}{3}\right)$**.

The graph is shown in the figure on the right.

(2) $y=x^2-8x=x^2-8x+16-16=(x-4)^2-16$ Therefore, the vertex is $(4,\ -16)$.

$y=-\frac{1}{2}x^2+ax-3b=-\frac{1}{2}(x^2-2ax)-3b=-\frac{1}{2}(x^2-2ax+a^2-a^2)-3b$

$=-\frac{1}{2}\{(x-a)^2-a^2\}-3b=-\frac{1}{2}(x-a)^2+\frac{1}{2}a^2-3b$

Therefore, since the vertex is $\left(a,\ \frac{1}{2}a^2-3b\right)$.

$$\begin{cases} 4=a & \cdots\cdots ① \\ -16=\frac{1}{2}a^2-3b & \cdots\cdots ② \end{cases}$$

Substituting ① into ②, we get $-16=\frac{1}{2}\cdot 4^2-3b=8-3b$. Solve this and we get $b=8$.

Therefore **$a=4$, $b=8$.**

5

(1) How much parallel shift of the parabola $y=-2x^2-14x-13$ will overlap the parabola $y=-2x^2+8x+7$?

(2) When the graph of the quadratic function $y=x^2+ax+4$ is translated by 2 along the x-axis direction to form the graph of the quadratic function $y=x^2-9x+b$, find the values of the constants a and b.

(3) Fill in the following blanks.

The graph of the quadratic function $y=x^2$ was translated by [(a)] along the x-axis direction and translated by [(b)] along the y-axis direction, and then symmetrically shifted with respect to [(c)], yields the graph of the quadratic function $y=-x^2-2x-2$.

solution

(1) $y=-2x^2-14x-13=-2(x^2+7x)-13=-2\left(x^2+7x+\frac{49}{4}-\frac{49}{4}\right)-13$

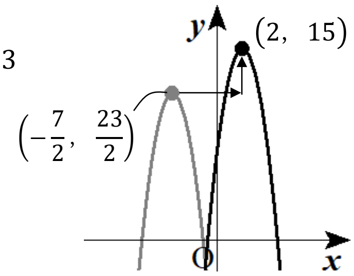
$=-2\left\{\left(x+\frac{7}{2}\right)^2-\frac{49}{4}\right\}-13=-2\left(x+\frac{7}{2}\right)^2+\frac{49}{2}-13$

$=-2\left(x+\frac{7}{2}\right)^2+\frac{23}{2}$, so the vertex is $\left(-\frac{7}{2},\ \frac{23}{2}\right)$.

$y=-2x^2+8x+7=-2(x^2-4x)+7=-2(x^2-4x+4-4)+7$

$=-2\{(x-2)^2-4\}+7=-2(x-2)^2+8+7=-2(x-2)^2+15$,

so the vertex is $(2,\ 15)$.



$2-\left(-\frac{7}{2}\right)=\frac{11}{2}$,

$15-\frac{23}{2}=\frac{7}{2}$

Therefore, $\frac{11}{2}$ **in the x-axis direction and $\frac{7}{2}$ parallel shift in the y-axis direction will result in overlap**.

(2) $y=x^2+ax+4=\left(x+\frac{a}{2}\right)^2-\frac{a^2}{4}+4$, so the vertex is $\left(-\frac{a}{2},\ -\frac{a^2}{4}+4\right)$.

$y=x^2-9x+b=\left(x-\frac{9}{2}\right)^2-\frac{81}{4}+b$, so the vertex is $\left(\frac{9}{2},\ -\frac{81}{4}+b\right)$.

From the coordinates of each vertex, we get $\begin{cases}-\frac{a}{2}+2=\frac{9}{2} & \cdots\cdots ① \\ -\frac{a^2}{4}+4=-\frac{81}{4}+b & \cdots\cdots ②\end{cases}$.

From ①, $-\frac{a}{2}=\frac{5}{2}$, therefore $a=-5$.

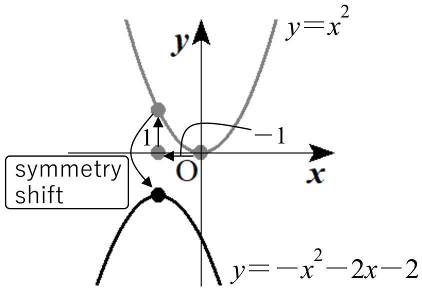
Substituting $a=-5$ for ②, $-\frac{(-5)^2}{4}+4=-\frac{81}{4}+b$. From this, $b=-\frac{25}{4}+4+\frac{81}{4}=18$.

From the above, $\boldsymbol{a=-5,\ b=18.}$

(3) $y=-x^2-2x-2=-(x^2+2x)-2=-\{(x+1)^2-1\}-2=-(x+1)^2+1-2=-(x+1)^2-1$

and so the vertex is $(-1,\ -1)$.

Thus, if the graph of the quadratic function $y=x^2$ is translated along the x-axis direction by **(a) -1** and along the y-axis direction by **(b) 1**, and then symmetrically shifted about the **(c) x-axis**, the expression for the graph becomes $y=-x^2-2x-2$.



〈Note〉 **(a) 1 , (b) 1 , and (c) origin** are also correct.

(a) 1 , (b) -1 , and (c) y-axis are incorrect because the direction of the graph must be opposite.

6

(1) Find the maximum and minimum values of the function $y=x^2+x+2$ $(-1\leqq x\leqq 1)$.

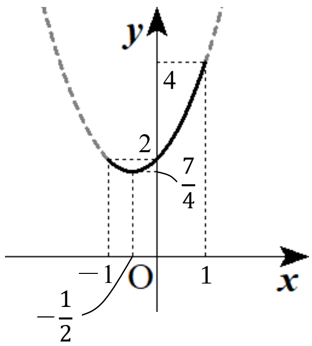
(2) Find the minimum value of the function $y=x^2-4x$ $(a\leqq x\leqq a+1)$ with constant a in the following three cases.

① $a<1$ ② $1\leqq a\leqq 2$ ③ $2<a$

solution

(1) $y=x^2+x+2=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+2$

$=\left(x+\frac{1}{2}\right)^2+\frac{7}{4}$



Since the domain of definition is $-1\leqq x\leqq 1$,

from the graph on the right,

the maximum value is 4 when $x=1$,

and the minimum value is $\frac{7}{4}$ when $x=-\frac{1}{2}$.

(2) $y=x^2-4x=(x-2)^2-4$ $(a\leqq x\leqq a+1)$

This graph is convex at the bottom, has a fixed vertex, and the domain of definition moves.

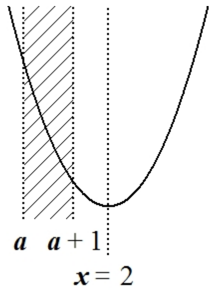
① When $a<1$

The axis is $x=2$ and $a+1<2$ when $a<1$, so the axis is right-outer the domain of definition $a\leqq x\leqq a+1$.

Therefore, **it is minimum at $x=a+1$, and the minimum value is**

$(a+1)^2-4(a+1)=a^2+2a+1-4a-4$

$=\boldsymbol{a^2-2a-3}$ **.**



② When $1\leqq a\leqq 2$

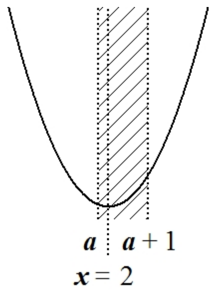
Substituting the axis $x=2$ into the domain of definition $a\leqq x\leqq a+1$,

so $a\leqq 2\leqq a+1$.

That is, from $\begin{cases} a\leqq 2 \\ 2\leqq a+1 \end{cases}$, we obtain $1\leqq a\leqq 2$.

Therefore, the axis $x=2$ is inside the domain of definition $a\leqq x\leqq a+1$.

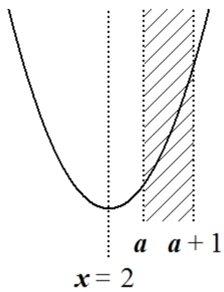
Therefore, **it is minimum is at $x=2$, and the minimum value is -4 .**



③ When $2<a$

The axis $x=2$ is outside the left of the domain of definition $a\leqq x\leqq a+1$.

Therefore, **it is minimum at $x=a$, and the minimum value is a^2-4a .**



7

Find a quadratic function that satisfies the following conditions.

(1) Through 3 points (2, 0), (1, 1), (3, 5).

(2) Tangent to the x-axis and passing through two points (1, 1) and (4, 4).

solution

(1) Substitute $x=2$, $y=0$ and $x=1$, $y=1$ and $x=3$, $y=5$ for $y=ax^2+bx+c$ to form a simultaneous equation.

$$\begin{cases} 0 = 4a + 2b + c & \cdots\cdots ① \\ 1 = a + b + c & \cdots\cdots ② \\ 5 = 9a + 3b + c & \cdots\cdots ③ \end{cases}.$$

From ①−②, ③−①

$$\begin{cases} -1 = 3a + b \\ 5 = 5a + b \end{cases}.$$ Solving this simultaneous equation yields $a = 3$, $b = -10$.

Substituting $a=3$ and $b=-10$ for ②, we obtain $c=8$.

From the above, $\boldsymbol{y=3x^2-10x+8}$.

(2) If it is tangent to the x-axis, the y-coordinate of the vertex is 0.

Therefore, the quadratic function to be obtained can be expressed as

$y=a(x-p)^2$.

Substitute $x=1$, $y=1$ and $x=4$, $y=4$ into this to form a simultaneous equation.

$$\begin{cases} 1 = a(1-p)^2 & \cdots\cdots ① \\ 4 = a(4-p)^2 & \cdots\cdots ② \end{cases}$$

Expanding the right-hand side, we get $\begin{cases} 1 = a - 2ap + ap^2 & \cdots\cdots ①' \\ 4 = 16a - 8ap + ap^2 & \cdots\cdots ②' \end{cases}$.

From ②′ −①′ $3=15a-6ap$

$1=5a-2ap$

$a(5-2p) = 1$ and $5-2p \neq 0$, so $a = \dfrac{1}{5-2p}$.

Substituting this into ①, $1 = \dfrac{1}{5-2p}(1-p)^2$.

Since $5-2p\neq 0$, multiplying both sides by $5-2p$, $5-2p=1-2p+p^2$.

To summarize, we have $p^2-4=0$. Solving for this, $p=\pm 2$.

When $p = 2$, we have $a = 1$ from ①. When $p = -2$, we have $a = \dfrac{1}{9}$ from ①.

Therefore $\boldsymbol{y = (x-2)^2}$, $\boldsymbol{y = \frac{1}{9}(x+2)^2}$, that is $\boldsymbol{y = x^2 - 4x + 4}$, $\boldsymbol{y = \frac{1}{9}x^2 + \frac{4}{9}x + \frac{4}{9}}$.

8

Solve the following quadratic equations.

(1) $x^2-10x+24=0$ (2) $14x^2+29x-15=0$

(3) $x^2+5x+5=0$ (4) $x^2-6x-6=0$

solution

(1) $x^2-10x+24=0$

Factorize the left-hand side to get $(x-4)(x-6)=0$.

Therefore, $x-4=0$ or $x-6=0$.

Thus, $\boldsymbol{x=4,\ 6}$.

(2) $14x^2+29x-15=0$

Factorize the left-hand side to get

$$(2x+5)(7x-3)=0.$$

2	5	→	35
7	−3	→	−6
			29

Therefore, $2x+5=0$ or $7x-3=0$.

Thus, $\boldsymbol{x=-\frac{5}{2},\ \frac{3}{7}}$.

(3) $x^2+5x+5=0$

By the solution formula of quadratic equation we obtain $\boldsymbol{x}=\frac{-5\pm\sqrt{5^2-4\cdot1\cdot5}}{2\cdot1}=\boldsymbol{\frac{-5\pm\sqrt{5}}{2}}$.

(4) $x^2-6x-6=0$

This quadratic equation can be viewed as $x^2+2\cdot(-3)x-6=0$, so

$$\boldsymbol{x}=\frac{-(-3)\pm\sqrt{(-3)^2-1\cdot(-6)}}{1}=\boldsymbol{3\pm\sqrt{15}}.$$

9

(1) Find the number of real solutions to the following quadratic equations.

① $-2x^2+6x-\frac{9}{2}=0$ ② $x^2-\frac{9}{2}x+5=0$

(2) When the quadratic equation $x^2-mx+m+3=0$ has multiple solution, find the value of the constant m. Also, find the multiple solution of the quadratic equation at that time.

solution

(1) Let D be the discriminant for the given quadratic equations.

① $D=6^2-4\cdot(-2)\cdot\left(-\frac{9}{2}\right)=36-36=0$

Since $D=0$, the number of real solutions is **1**.

② $D=\left(-\frac{9}{2}\right)^2-4\cdot1\cdot5=\frac{81}{4}-20=\frac{1}{4}$

Since $D>0$, the number of real solutions is **2**.

(2) Let D be the discriminant of the given quadratic equation, and we obtain

$$D=(-m)^2-4\cdot1\cdot(m+3)=m^2-4m-12=(m+2)(m-6).$$

The condition for having a multiple solution is that $D=0$ holds.

Therefore, $(m+2)(m-6)=0$. Solving for this, $m=-2,\ 6$.

When $m=-2$, the quadratic equation is $x^2+2x+1=0$. Solving for this, $x=-1$.

When $m=6$, the quadratic equation is $x^2-6x+9=0$. Solving for this, $x=3$.

Thus, **when $m=-2$, the multiple solution is $x=-1$, and when $m=6$, the multiple solution is $x=3$.**

１０

How does the number of common points by the graph of the quadratic function $y=-x^2+4x+2k$ with the x-axis vary with the value of the constant k ?

solution

Let D be the discriminant of the quadratic equation $-x^2+4x+2k=0$.

$$D=4^2-4\cdot(-1)\cdot 2k=16+8k$$

(i) When we have two different common points, $D>0$, so $16+8k>0$ i.e. $k>-2$.

(ii) When they are tangent at one point, $D=0$, so $16+8k=0$, i.e. $k=-2$.

(iii) When there is no shared point, $D<0$, then $16+8k<0$, i.e. $k<-2$.

From (i), (ii), and (iii), $\begin{cases}\textbf{When } k>-2\textbf{, there are 2 .}\\ \textbf{When } k=-2\textbf{, there is 1 .}\\ \textbf{When } k<-2\textbf{, the number is 0 .}\end{cases}$

1 1

(1) Solve the following quadratic inequalities.

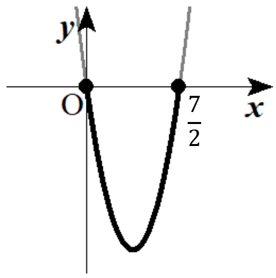
① $2x^2 \leqq 7x$　　　② $x^2 - x + \frac{1}{4} > 0$

(2) Solve the simultaneous inequalities $\begin{cases} x^2 + 2x - 3 \leqq 0 \\ x^2 + x - 1 > 0 \end{cases}$.

solution

(1) ① $2x^2 \leqq 7x \quad \Rightarrow \quad 2x^2 - 7x \leqq 0$

The solution of $2x^2 - 7x = 0$ is $x = 0,\ \frac{7}{2}$ from $2x^2 - 7x = x(2x - 7) = 0$.



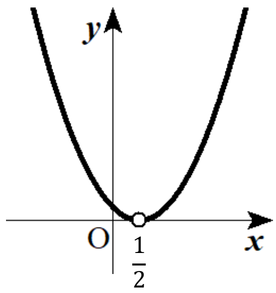
Therefore, the solution of the inequality is $\mathbf{0 \leqq x \leqq \frac{7}{2}}$

from the figure on the right.

② $x^2 - x + \frac{1}{4} > 0$

The solution of $x^2 - x + \frac{1}{4} = 0$ is $x = \frac{1}{2}$

from $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 = 0$.



Therefore, the solution of the inequality is

All real numbers except $\mathbf{\frac{1}{2}}$ $\left(\text{or } \boldsymbol{x < \frac{1}{2},\ \frac{1}{2} < x}\right)$

from the figure on the right.

(2) ・$x^2 + 2x - 3 \leqq 0$

The solution of $x^2 + 2x - 3 = 0$ is $x = -3,\ 1$ from $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$.

Therefore, the solution of the inequality is $\mathbf{-3 \leqq x \leqq 1}$.

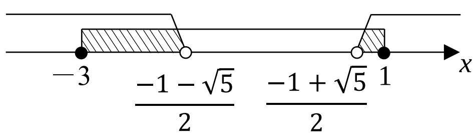
・$x^2 + x - 1 > 0$

The solution of $x^2 + x - 1 = 0$ is $x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{5}}{2}$.

Therefore, the solution of the inequality is $x < \frac{-1 - \sqrt{5}}{2},\ \frac{-1 + \sqrt{5}}{2} < x$.

From the number line on the right, the solution of the simultaneous inequality is

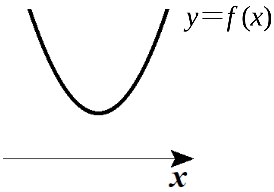
$\mathbf{-3 \leqq x < \frac{-1 - \sqrt{5}}{2},\ \frac{-1 + \sqrt{5}}{2} < x \leqq 1}$.



12

Find the range of values of the constant k such that the quadratic inequality $x^2+(k-2)x-k+10>0$ holds for all real numbers x .

solution



If $f(x)=x^2+(k-2)x-k+10$, the coefficient of x^2 in $f(x)$ is positive, so the graph of the quadratic function $y=f(x)$ is convex downward .
Therefore, the condition for $f(x)>0$ to hold for all real numbers x is that the graph of $y=f(x)$ is always above the x-axis, i.e., the graph of $y=f(x)$ has no common points with the x-axis .
Thus, let D be the discriminant of the quadratic equation $f(x)=0$. It is sufficient if $D<0$.
where $D=(k-2)^2-4\cdot1\cdot(-k+10)=k^2-4k+4+4k-40=k^2-36=(k+6)(k-6)$,
so $(k+6)(k-6)<0$ from $D<0$.
Solving for this, we get $\boldsymbol{-6<k<6}$ **.**

13

Determine the range of values of the constant m so that the graph of the quadratic function $y=x^2-(m+2)x+5$ has two different common points on the positive part of the x-axis.

solution

For the graph of the quadratic function $y=x^2-(m+2)x+5$, let $f(x)=x^2-(m+2)x+5$ and the discriminant D of the quadratic equation $f(x)=0$.

$$D=\{-(m+2)\}^2-4\cdot 1\cdot 5=m^2+4m+4-20=m^2+4m-16$$

$D>0$, (position of the axis)>0, and $f(0)>0$.

(i) $D>0$ i.e. $m^2+4m-16>0$

The solution to $m^2+4m-16=0$ is

$$m=\frac{-4\pm\sqrt{4^2-4\cdot 1\cdot(-16)}}{2\cdot 1}=\frac{-4\pm\sqrt{16+64}}{2}=\frac{-4\pm 4\sqrt{5}}{2}=-2\pm 2\sqrt{5}\,.$$

Therefore, $m<-2-2\sqrt{5},\quad -2+2\sqrt{5}<m$.

(ii) (position of the axis)>0

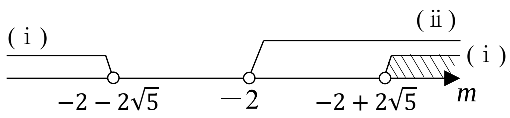
From $x^2-(m+2)x+5=\left(x-\frac{m+2}{2}\right)^2-\left(\frac{m+2}{2}\right)^2+5$, the axis is straight line $x=\frac{m+2}{2}$.

From this, $\frac{m+2}{2}>0$. Therefore, $m>-2$.

(iii) $f(0)>0$

From $f(0)=5$, $f(0)>0$ is always satisfied .

From the above, it is $\boldsymbol{m>-2+2\sqrt{5}}$ from the number line on the right .



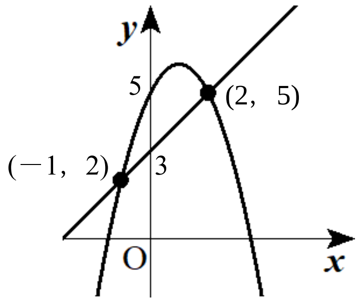
Study

(1) Find the coordinates of the common point by the parabola $y=-x^2+2x+5$ and the line $y=x+3$.

(2) Let b be a real number . Find the value of the constant b such that the parabola $y=x^2-2x-2$ and the line $y=2x+b$ are tangent .

solution

(1) Find the solution of the quadratic equation $x+3=-x^2+2x+5$ obtained by eliminating y .



$x+3=-x^2+2x+5 \quad \Rightarrow \quad x^2-x-2=0 \quad \Rightarrow \quad (x+1)(x-2)=0$

Solving for this, we get $x=-1,\ 2$.

$y=(-1)+3=2$ when $x=-1$.

$y=2+3=5$ when $x=2$.

Therefore, the coordinates of the common point to be sought are

(−1, 2) and (2, 5) .

(2) The number of real solutions to the quadratic equation $2x+b=x^2-2x-2$ obtained by eliminating y should be one.

$2x+b=x^2-2x-2 \quad \Rightarrow \quad x^2-4x-2-b=0$

Let D be the discriminant equation of the quadratic equation $x^2-4x-2-b=0$.

$$D=(-4)^2-4\cdot 1\cdot(-2-b)=16+8+4b=24+4b$$

The intent of the problem is satisfied when $D=0$. Therefore, $\boldsymbol{b=-6}$