Expansion and factorization of expressions

1

(1) Arrange the following expressions in descending order for the letters in [ ] and answer the order and constant terms when focusing on the letters in [ ].

① $2a^2+3+a^4+2a^4+3a^2+a^6$ [a]

② $x^2+y^2+z^2+xy+yz+zx$ [z]

(2) If $A=x^2+2ax+2$ and $B=a^2-3ax+1$, calculate the following.

① $3A+2B$

② $A-\{2B+3(A-2B)\}$

solution

(1) ① $2a^2+3+a^4+2a^4+3a^2+a^6=\boldsymbol{a^6+3a^4+5a^2+3}$

The order is 6 and the constant term is 3.

② $x^2+y^2+z^2+xy+yz+zx=\boldsymbol{z^2+(x+y)z+x^2+y^2+xy}$

The order is 2, the constant term is $\boldsymbol{x^2+y^2+xy}$.

(2) ① $3A+2B=3(x^2+2ax+2)+2(a^2-3ax+1)=3x^2+6ax+6+2a^2-6ax+2$

$=\boldsymbol{3x^2+2a^2+8}$

② $A-\{2B+3(A-2B)\}=A-(2B+3A-6B)=A-2B-3A+6B=-2A+4B$

$=-2(x^2+2ax+2)+4(a^2-3ax+1)=-2x^2-4ax-4+4a^2-12ax+4$

$=\boldsymbol{-2x^2-16ax+4a^2}$

2

(1) Calculate the following expressions.

① $(-2a^2b)^3$

② $x^2y^3\times(-xy^2z)^2$

(2) Expand the following expressions.

① $(x^2-x-1)(2x+1)$

② $(a+b+1)(2a-3b-1)$

solution

(1) ① $(-2a^2b)^3=(-2)^3a^{2\times3}b^3=\boldsymbol{-8a^6b^3}$

② $x^2y^3\times(-xy^2z)^2=x^2y^3\times(-1)^2x^2y^{2\times2}z^2=1\times x^{2+2}\times y^{3+4}\times z^2=\boldsymbol{x^4y^7z^2}$

(2) ① $(x^2-x-1)(2x+1)=(x^2-x-1)\cdot2x+(x^2-x-1)\cdot1=2x^3-2x^2-2x+x^2-x-1$

$=\boldsymbol{2x^3-x^2-3x-1}$

② $(a+b+1)(2a-3b-1)=a(2a-3b-1)+b(2a-3b-1)+1\cdot(2a-3b-1)$

$=2a^2-3ab-a+2ab-3b^2-b+2a-3b-1$

$=\boldsymbol{2a^2-ab-3b^2+a-4b-1}$

3 Expand the following expressions.

(1) $(a-2b)^2$ (2) $(3+2x)(3-2x)$

(3) $(a-5)(a+7)$ (4) $(5x-4y)(3x+2y)$

solution

(1) $(a-2b)^2=a^2-2\cdot a\cdot 2b+(2b)^2=\mathbf{a^2-4ab+4b^2}$

(2) $(3+2x)(3-2x)=3^2-(2x)^2=\mathbf{9-4x^2}$

(3) $(a-5)(a+7)=a^2+(-5+7)a-5\cdot 7=\mathbf{a^2+2a-35}$

(4) $(5x-4y)(3x+2y)=(5\cdot 3)x^2+\{5\cdot 2y+(-4y)\cdot 3\}x-4y\cdot 2y=\mathbf{15x^2-2xy-8y^2}$

4 Expand the following expressions.

(1) $(x^2+x+1)^2$ (2) $(4a^2+1)(2a+1)(2a-1)$

solution

(1) If $x^2+x=A$, then $(x^2+x+1)^2=(A+1)^2=A^2+2A+1=(x^2+x)^2+2(x^2+x)+1$

$$=x^4+2x^3+x^2+2x^2+2x+1=\mathbf{x^4+2x^3+3x^2+2x+1}$$

Alternative solution

Substituting $a=x^2$, $b=x$, $c=1$ for $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$,

$$(x^2+x+1)^2=(x^2)^2+x^2+1^2+2\cdot x^2\cdot x+2\cdot x\cdot 1+2\cdot 1\cdot x^2$$

$$=x^4+x^2+1+2x^3+2x+2x^2=x^4+2x^3+3x^2+2x+1$$

(2) $(4a^2+1)(2a+1)(2a-1)=(4a^2+1)\{(2a)^2-1^2\}=(4a^2+1)(4a^2-1)=(4a^2)^2-1^2$

$$=\mathbf{16a^4-1}$$

5 Factorize the following expressions.

(1) $3ax^2-6a^2b$ (2) $16a^2+8a+1$ (3) $x^2-x+\frac{1}{4}$

(4) $64x^2-25y^2$ (5) $a^2+3ab-10b^2$ (6) $3x^2-12$

solution

(1) $3ax^2-6a^2b=3a\cdot x^2-3a\cdot 2ab=\mathbf{3a(x^2-2ab)}$

(2) $16a^2+8a+1=(4a)^2+2\cdot 4a\cdot 1+1^2=\mathbf{(4a+1)^2}$

(3) $x^2-x+\frac{1}{4}=x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2=\left(\mathbf{x-\frac{1}{2}}\right)^2$

(4) $64x^2-25y^2=(8x)^2-(5y)^2=\mathbf{(8x+5y)(8x-5y)}$

(5) $a^2+3ab-10b^2=a^2+(5b-2b)a+5b\cdot(-2b)=\mathbf{(a+5b)(a-2b)}$

(6) $3x^2-12=3(x^2-4)=3(x^2-2^2)=\mathbf{3(x+2)(x-2)}$

6 Factorize the following expressions.

(1) $3x^2+5x-2$　　(2) $4a^2+8a+3$

(3) $6x^2+xy-2y^2$　　(4) $8a^2-14ab-15b^2$

solution

(1) $3x^2+5x-2=\mathbf{(x+2)(3x-1)}$

1	2	→	6
3	-1	→	-1
			5

(2) $4a^2+8a+3=\mathbf{(2a+1)(2a+3)}$

2	1	→	2
2	3	→	6
			8

(3) $6x^2+xy-2y^2=\mathbf{(2x-y)(3x+2y)}$

2	$-y$	→	$-3y$
3	$2y$	→	$4y$
			y

(4) $8a^2-14ab-15b^2=\mathbf{(2a-5b)(4a+3b)}$

2	$-5b$	→	$-20b$
4	$3b$	→	$6b$
			$-14b$

7 Factorize the following expressions.

(1) $(x+y+1)(x+y+2)-6$　　(2) $4a^2-9b^2+6bc-c^2$

solution

(1) If $x+y=A$,then

$$(x+y+1)(x+y+2)-6=(A+1)(A+2)-6=A^2+3A+2-6=A^2+3A-4$$
$$=(A+4)(A-1)$$
$$=\mathbf{(x+y+4)(x+y-1)}$$

(2) From $4a^2-9b^2+6bc-c^2=4a^2-(9b^2-6bc+c^2)=4a^2-\{(3b)^2-2\cdot 3b\cdot c+c^2\}=4a^2-(3b-c)^2$,

if $3b-c=A$,then

$$4a^2-9b^2+6bc-c^2=4a^2-A^2=(2a)^2-A^2=(2a+A)(2a-A)$$
$$=\{2a+(3b-c)\}\{2a-(3b-c)\}$$
$$=\mathbf{(2a+3b-c)(2a-3b+c)}$$

8 Factorize the following expressions.

(1) $x^2-2xy+3x-4y+2$ (2) $6ab+4a-3b-2$

(3) $2x^2+3xy-2y^2+x+2y$ (4) $2x^2+7xy+3y^2-5x-10y+3$

solution

(1) $x^2-2xy+3x-4y+2=(-2x-4)y+x^2+3x+2=-2(x+2)y+(x+1)(x+2)$

$=(x+2)\{-2y+(x+1)\}=\mathbf{(x+2)(x-2y+1)}$

(2) $6ab+4a-3b-2=2a(3b+2)-(3b+2)$

$=\mathbf{(3b+2)(2a-1)}$

Alternative solution $6ab+4a-3b-2=3b(2a-1)+2(2a-1)=(2a-1)(3b+2)$

(3) $2x^2+3xy-2y^2+x+2y=2x^2+(3y+1)x-2y^2+2y$

$=2x^2+(3y+1)x-2y(y-1)$

$=(x+2y)\{2x-(y-1)\}$

$=\mathbf{(x+2y)(2x-y+1)}$

1	$2y$	→	$4y$
2	$-(y-1)$	→	$-y+1$
			$3y+1$

〈Note〉 The same factorization can be done by arranging y in descending order.

Alternative solution First factorize the quadratic expression.

$2x^2+3xy-2y^2+x+2y=(\underline{x+2y})(2x-y)+\underline{x+2y}$

$=(x+2y)(2x-y+1)$

1	$2y$	→	$4y$
2	$-y$	→	$-y$
			$3y$

(4) $2x^2+7xy+3y^2-5x-10y+3=2x^2+(7y-5)x+3y^2-10y+3$

1	-3	→	-9
3	-1	→	-1
			-10

$=2x^2+(7y-5)x+(y-3)(3y-1)$

$=\{x+(3y-1)\}\{2x+(y-3)\}$

$=\mathbf{(x+3y-1)(2x+y-3)}$

1	$(3y-1)$	→	$6y-2$
2	$(y-3)$	→	$y-3$
			$7y-5$

〈Note〉 The same factorization can be done by arranging y in descending order.

Alternative solution First factorize the quadratic expression.

$2x^2+7xy+3y^2-5x-10y+3=(x+3y)(2x+y)-5x-10y+3$

1	$3y$	→	$6y$
2	y	→	y
			$7y$

In this case, the quadratic expression for t

$(x+3y)(2x+y)t^2+(-5x-10y)t+3$

and factorize it by using the cross multiplication

$(x+3y)(2x+y)t^2+(-5x-10y)t+3$

$=\{(x+3y)t-1\}\{(2x+y)t-3\}$

$(x+3y)$	-1	→	$-2x-y$
$(2x+y)$	-3	→	$-3x-9y$
			$-5x-10y$

Where, substituting $t=1$

$(x+3y)(2x+y)-5x-10y+3=(x+3y-1)(2x+y-3)$

Study 1

(1) Expand the following expressions.

① $(3x-1)^3$　　② $(4a+3b)(16a^2-12ab+9b^2)$

(2) Factorize the following equation.

① $1-a^3$　　② $1000x^3+y^3$

solution

(1) ① $(3x-1)^3=(3x)^3-3\cdot(3x)^2\cdot 1+3\cdot 3x\cdot 1^2-1^3=\mathbf{27x^3-27x^2+9x-1}$

② $(4a+3b)(16a^2-12ab+9b^2)=(4a+3b)\{(4a)^2-4a\cdot 3b+(3b)^2\}=(4a)^3+(3b)^3$

$=\mathbf{64a^3+27b^3}$

(2) ① $1-a^3=1^3-a^3=(1-a)(1^2+1\cdot a+a^2)=\mathbf{(1-a)(1+a+a^2)}$

② $1000x^3+y^3=(10x)^3+y^3=(10x+y)\{(10x)^2-10x\cdot y+y^2\}$

$=\mathbf{(10x+y)(100x^2-10xy+y^2)}$

Study 2 Factorize the following equation.

(1) x^4-1　　(2) x^4-2x^2-8

(3) x^4+4　　(4) x^4-3x^2+1

solution

(1) If $x^2=X$, then $x^4-1=(x^2)^2-1=X^2-1=(X+1)(X-1)=(x^2+1)(x^2-1)$

$=\mathbf{(x^2+1)(x+1)(x-1)}$

(2) If $x^2=X$, then $x^4-2x^2-8=(x^2)^2-2x^2-8=X^2-2X-8=(X+2)(X-4)=(x^2+2)(x^2-4)$

$=\mathbf{(x^2+2)(x+2)(x-2)}$

(3) From $(x^2+2)^2=x^4+4x^2+4$,

$$x^4+4=x^4+4x^2+4-4x^2=(x^2+2)^2-4x^2$$

and can be transformed. From this it follows that

$$x^4+4=(x^2+2)^2-4x^2=\{(x^2+2)+2x\}\{(x^2+2)-2x\}$$

$$=\mathbf{(x^2+2x+2)(x^2-2x+2)}$$

(4) From $(x^2-1)^2=x^4-2x^2+1$,

$$x^4-3x^2+1=x^4-2x^2+1-x^2=(x^2-1)^2-x^2$$

and can be transformed. From this it follows that

$$x^4-3x^2+1=(x^2-1)^2-x^2=\{(x^2-1)+x\}\{(x^2-1)-x\}$$

$$=\mathbf{(x^2+x-1)(x^2-x-1)}$$